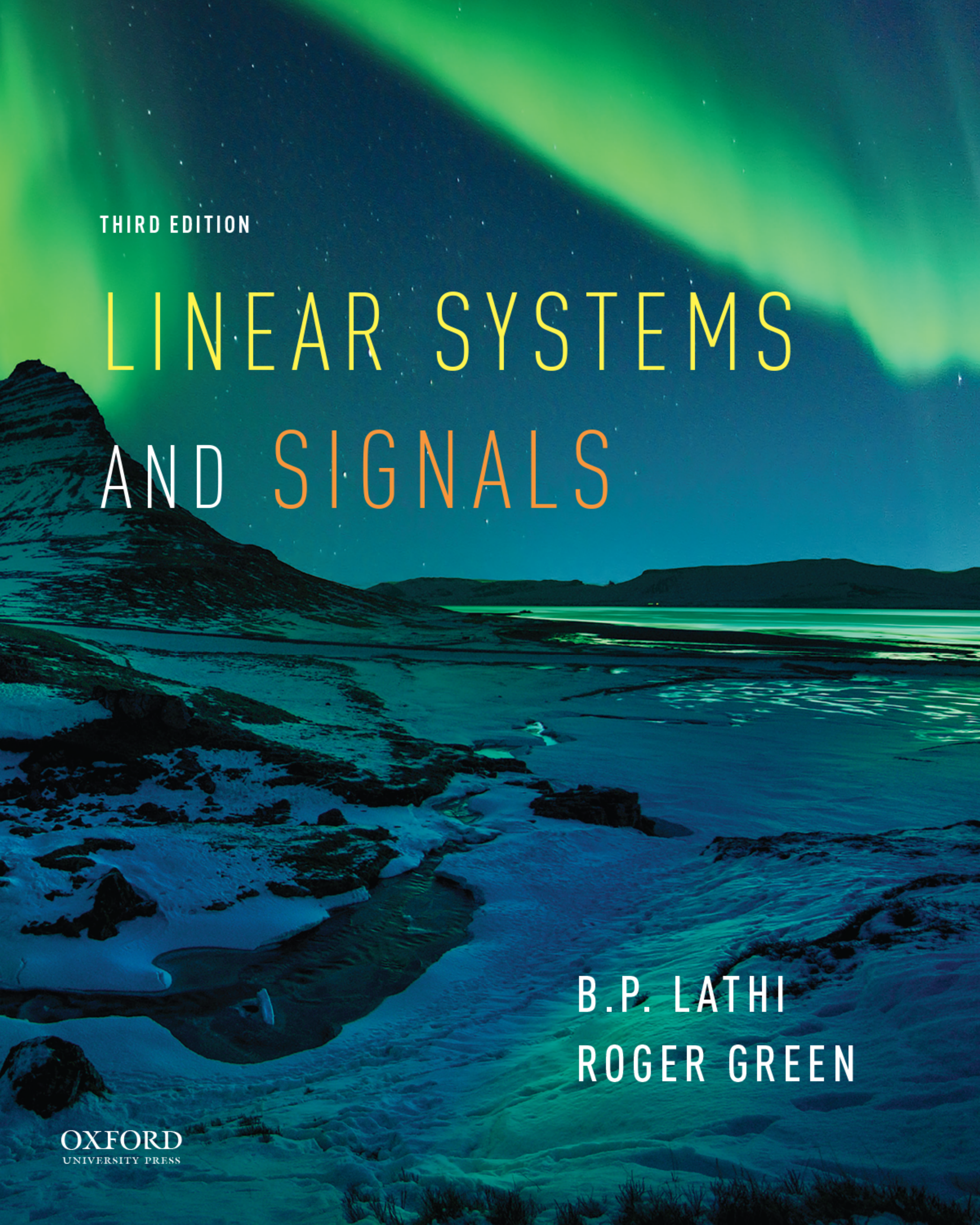

THIRD EDITION

LINEAR SYSTEMS AND SIGNALS

B.P. LATHI
ROGER GREEN

OXFORD
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LINEAR SYSTEMS AND SIGNALS

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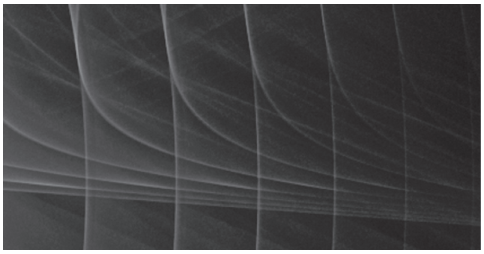

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PREFACE

This book, *Linear Systems and Signals,* presents a comprehensive treatment of signals and linear systems at an introductory level. Following our preferred style, it emphasizes a physical appreciation of concepts through heuristic reasoning and the use of metaphors, analogies, and creative explanations. Such an approach is much different from a purely deductive technique that uses mere mathematical manipulation of symbols. There is a temptation to treat engineering subjects as a branch of applied mathematics. Such an approach is a perfect match to the public image of engineering as a dry and dull discipline. It ignores the physical meaning behind various derivations and deprives students of intuitive grasp and the enjoyable experience of logical uncovering of the subject matter. In this book, we use mathematics not so much to prove axiomatic theory as to support and enhance physical and intuitive understanding. Wherever possible, theoretical results are interpreted heuristically and are enhanced by carefully chosen examples and analogies.

This third edition, which closely follows the organization of the second edition, has been refined in many ways. Discussions are streamlined, adding or trimming material as needed. Equation, example, and section labeling is simplified and improved. Computer examples are fully updated to reflect the most current version of MATLAB. Hundreds of added problems provide new opportunities to learn and understand topics. We have taken special care to improve the text without the topic creep and bloat that commonly occurs with each new edition of a text.

NOTABLE FEATURES

The notable features of this book include the following.

1. Intuitive and heuristic understanding of the concepts and physical meaning of mathematical results are emphasized throughout. Such an approach not only leads to deeper appreciation and easier comprehension of the concepts, but also makes learning enjoyable for students.
2. Often, students lack an adequate background in basic material such as complex numbers, sinusoids, hand-sketching of functions, Cramer's rule, partial fraction expansion, and matrix algebra. We include a background chapter that addresses these basic and pervasive topics in electrical engineering. Response by students has been unanimously enthusiastic.
3. There are hundreds of worked examples in addition to drills (usually with answers) for students to test their understanding. Additionally, there are over 900 end-of-chapter problems of varying difficulty.
4. Modern electrical engineering practice requires the use of computer calculation and simulation, most often using the software package MATLAB. Thus, we integrate

MATLAB into many of the worked examples throughout the book. Additionally, each chapter concludes with a section devoted to learning and using MATLAB in the context and support of book topics. Problem sets also contain numerous computer problems.
5. The discrete-time and continuous-time systems may be treated in sequence, or they may be integrated by using a parallel approach.
6. The summary at the end of each chapter proves helpful to students in summing up essential developments in the chapter.
7. There are several historical notes to enhance students' interest in the subject. This information introduces students to the historical background that influenced the development of electrical engineering.

ORGANIZATION

The book may be conceived as divided into five parts:

1. Introduction (Chs. B and 1).
2. Time-domain analysis of linear time-invariant (LTI) systems (Chs. 2 and 3).
3. Frequency-domain (transform) analysis of LTI systems (Chs. 4 and 5).
4. Signal analysis (Chs. 6, 7, 8, and 9).
5. State-space analysis of LTI systems (Ch. 10).

The organization of the book permits much flexibility in teaching the continuous-time and discrete-time concepts. The natural sequence of chapters is meant to integrate continuous-time and discrete-time analysis. It is also possible to use a sequential approach in which all the continuous-time analysis is covered first (Chs. 1, 2, 4, 6, 7, and 8), followed by discrete-time analysis (Chs. 3, 5, and 9).

SUGGESTIONS FOR USING THIS BOOK

The book can be readily tailored for a variety of courses spanning 30 to 45 lecture hours. Most of the material in the first eight chapters can be covered at a brisk pace in about 45 hours. The book can also be used for a 30-lecture-hour course by covering only analog material (Chs. 1, 2, 4, 6, 7, and possibly selected topics in Ch. 8). Alternately, one can also select Chs. 1 to 5 for courses purely devoted to systems analysis or transform techniques. To treat continuous- and discrete-time systems by using an integrated (or parallel) approach, the appropriate sequence of chapters is 1, 2, 3, 4, 5, 6, 7, and 8. For a sequential approach, where the continuous-time analysis is followed by discrete-time analysis, the proper chapter sequence is 1, 2, 4, 6, 7, 8, 3, 5, and possibly 9 (depending on the time available).

MATLAB

MATLAB is a sophisticated language that serves as a powerful tool to better understand engineering topics, including control theory, filter design, and, of course, linear systems and signals. MATLAB's flexible programming structure promotes rapid development and analysis. Outstanding visualization capabilities provide unique insight into system behavior and signal character.

As with any language, learning MATLAB is incremental and requires practice. This book provides two levels of exposure to MATLAB. First, MATLAB is integrated into many examples throughout the text to reinforce concepts and perform various computations. These examples utilize standard MATLAB functions as well as functions from the control system, signal-processing, and symbolic math toolboxes. MATLAB has many more toolboxes available, but these three are commonly available in most engineering departments.

A second and deeper level of exposure to MATLAB is achieved by concluding each chapter with a separate MATLAB section. Taken together, these eleven sections provide a self-contained introduction to the MATLAB environment that allows even novice users to quickly gain MATLAB proficiency and competence. These sessions provide detailed instruction on how to use MATLAB to solve problems in linear systems and signals. Except for the very last chapter, special care has been taken to avoid the use of toolbox functions in the MATLAB sessions. Rather, readers are shown the process of developing their own code. In this way, those readers without toolbox access are not at a disadvantage. All of this book's MATLAB code is available for download at the OUP companion website www.oup.com/us/lathi.

CREDITS AND ACKNOWLEDGMENTS

The portraits of Gauss, Laplace, Heaviside, Fourier, and Michelson have been reprinted courtesy of the Smithsonian Institution Libraries. The likenesses of Cardano and Gibbs have been reprinted courtesy of the Library of Congress. The engraving of Napoleon has been reprinted courtesy of Bettmann/Corbis. The many fine cartoons throughout the text are the work of Joseph Coniglio, a former student of Dr. Lathi.

Many individuals have helped us in the preparation of this book, as well as its earlier editions. We are grateful to each and every one for helpful suggestions and comments. Book writing is an obsessively time-consuming activity, which causes much hardship for an author's family. We both are grateful to our families for their enormous but invisible sacrifices.

B. P. Lathi
R. A. Green

CHAPTER B

BACKGROUND

The topics discussed in this chapter are not entirely new to students taking this course. You have already studied many of these topics in earlier courses or are expected to know them from your previous training. Even so, this background material deserves a review because it is so pervasive in the area of signals and systems. Investing a little time in such a review will pay big dividends later. Furthermore, this material is useful not only for this course but also for several courses that follow. It will also be helpful later, as reference material in your professional career.

B.1 COMPLEX NUMBERS

Complex numbers are an extension of ordinary numbers and are an integral part of the modern number system. Complex numbers, particularly *imaginary numbers,* sometimes seem mysterious and unreal. This feeling of unreality derives from their unfamiliarity and novelty rather than their supposed nonexistence! Mathematicians blundered in calling these numbers "imaginary," for the term immediately prejudices perception. Had these numbers been called by some other name, they would have become demystified long ago, just as irrational numbers or negative numbers were. Many futile attempts have been made to ascribe some physical meaning to imaginary numbers. However, this effort is needless. In mathematics we assign symbols and operations any meaning we wish as long as internal consistency is maintained. The history of mathematics is full of entities that were unfamiliar and held in abhorrence until familiarity made them acceptable. This fact will become clear from the following historical note.

B.1-1 A Historical Note

Among early people the number system consisted only of natural numbers (positive integers) needed to express the number of children, cattle, and quivers of arrows. These people had no need for fractions. Whoever heard of two and one-half children or three and one-fourth cows!

However, with the advent of agriculture, people needed to measure continuously varying quantities, such as the length of a field and the weight of a quantity of butter. The number system, therefore, was extended to include fractions. The ancient Egyptians and Babylonians knew how to handle fractions, but *Pythagoras* discovered that some numbers (like the diagonal of a unit square) could not be expressed as a whole number or a fraction. Pythagoras, a number mystic, who regarded numbers as the essence and principle of all things in the universe, was so appalled at his discovery that he swore his followers to secrecy and imposed a death penalty for divulging this secret [1]. These numbers, however, were included in the number system by the time of Descartes, and they are now known as *irrational numbers*.

Until recently, *negative numbers* were not a part of the number system. The concept of negative numbers must have appeared absurd to early man. However, the medieval Hindus had a clear understanding of the significance of positive and negative numbers [2, 3]. They were also the first to recognize the existence of absolute negative quantities [4]. The works of *Bhaskar* (1114–1185) on arithmetic (*Līlāvatī*) and algebra (*Bījaganit*) not only use the decimal system but also give rules for dealing with negative quantities. Bhaskar recognized that positive numbers have two square roots [5]. Much later, in Europe, the men who developed the banking system that arose in Florence and Venice during the late Renaissance (fifteenth century) are credited with introducing a crude form of negative numbers. The seemingly absurd subtraction of 7 from 5 seemed reasonable when bankers began to allow their clients to draw seven gold ducats while their deposit stood at five. All that was necessary for this purpose was to write the difference, 2, on the debit side of a ledger [6].

Thus, the number system was once again broadened (generalized) to include negative numbers. The acceptance of negative numbers made it possible to solve equations such as $x+5=0$, which had no solution before. Yet for equations such as $x^2+1=0$, leading to $x^2=-1$, the solution could not be found in the real number system. It was therefore necessary to define a completely new kind of number with its square equal to -1. During the time of Descartes and Newton, imaginary (or complex) numbers came to be accepted as part of the number system, but they were still regarded as algebraic fiction. The Swiss mathematician *Leonhard Euler* introduced the notation i (for *imaginary*) around 1777 to represent $\sqrt{-1}$. Electrical engineers use the notation j instead of i to avoid confusion with the notation i often used for electrical current. Thus,

$$j^2=-1 \qquad \text{and} \qquad \sqrt{-1}=\pm j$$

This notation allows us to determine the square root of any negative number. For example,

$$\sqrt{-4}=\sqrt{4}\times\sqrt{-1}=\pm 2j$$

When imaginary numbers are included in the number system, the resulting numbers are called *complex numbers*.

Origins of Complex Numbers

Ironically (and contrary to popular belief), it was not the solution of a quadratic equation, such as $x^2+1=0$, but a cubic equation with real roots that made imaginary numbers plausible and acceptable to early mathematicians. They could dismiss $\sqrt{-1}$ as pure nonsense when it appeared as a solution to $x^2+1=0$ because this equation has no real solution. But in 1545, *Gerolamo Cardano* of Milan published *Ars Magna* (The Great Art), the most important algebraic work of the Renaissance. In this book, he gave a method of solving a general cubic equation in which a root of a negative number appeared in an intermediate step. According to his method, the solution to a third-order equation†

$$x^3 + ax + b = 0$$

is given by

$$x = \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} + \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}$$

For example, to find a solution of $x^3 + 6x - 20 = 0$, we substitute $a = 6, b = -20$ in the foregoing equation to obtain

$$x = \sqrt[3]{10 + \sqrt{108}} + \sqrt[3]{10 - \sqrt{108}} = \sqrt[3]{20.392} - \sqrt[3]{0.392} = 2$$

We can readily verify that 2 is indeed a solution of $x^3 + 6x - 20 = 0$. But when Cardano tried to solve the equation $x^3 - 15x - 4 = 0$ by this formula, his solution was

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

What was Cardano to make of this equation in the year 1545? In those days, negative numbers were themselves suspect, and a square root of a negative number was doubly preposterous! Today, we know that

$$(2 \pm j)^3 = 2 \pm j11 = 2 \pm \sqrt{-121}$$

Therefore, Cardano's formula gives

$$x = (2 + j) + (2 - j) = 4$$

We can readily verify that $x = 4$ is indeed a solution of $x^3 - 15x - 4 = 0$. Cardano tried to explain halfheartedly the presence of $\sqrt{-121}$ but ultimately dismissed the whole enterprise as being "as subtle as it is useless." A generation later, however, *Raphael Bombelli* (1526–1573), after examining Cardano's results, proposed acceptance of imaginary numbers as a necessary vehicle that would transport the mathematician from the *real* cubic equation to its *real* solution. In other words, although we begin and end with real numbers, we seem compelled to move into an unfamiliar world of imaginaries to complete our journey. To mathematicians of the day, this proposal seemed incredibly strange [7]. Yet they could not dismiss the idea of imaginary numbers so easily because this concept yielded the real solution of an equation. It took two more centuries for the full importance of complex numbers to become evident in the works of Euler, Gauss, and Cauchy. Still, Bombelli deserves credit for recognizing that such numbers have a role to play in algebra [7].

---

† This equation is known as the *depressed cubic* equation. A general cubic equation

$$y^3 + py^2 + qy + r = 0$$

can always be reduced to a depressed cubic form by substituting $y = x - (p/3)$. Therefore, any general cubic equation can be solved if we know the solution to the depressed cubic. The depressed cubic was independently solved, first by *Scipione del Ferro* (1465–1526) and then by *Niccolo Fontana* (1499–1557). The latter is better known in the history of mathematics as *Tartaglia* ("Stammerer"). Cardano learned the secret of the depressed cubic solution from Tartaglia. He then showed that by using the substitution $y = x - (p/3)$, a general cubic is reduced to a depressed cubic.

In 1799 the German mathematician *Karl Friedrich Gauss,* at the ripe age of 22, proved the fundamental theorem of algebra, namely that every algebraic equation in one unknown has a root in the form of a complex number. He showed that every equation of the nth order has exactly n solutions (roots), no more and no less. Gauss was also one of the first to give a coherent account of complex numbers and to interpret them as points in a complex plane. It is he who introduced the term *complex numbers* and paved the way for their general and systematic use. The number system was once again broadened or generalized to include imaginary numbers. Ordinary (or real) numbers became a special case of generalized (or complex) numbers.

The utility of complex numbers can be understood readily by an analogy with two neighboring countries X and Y, as illustrated in Fig. B.1. If we want to travel from City a to City b (both in

Gerolamo Cardano

Karl Friedrich Gauss

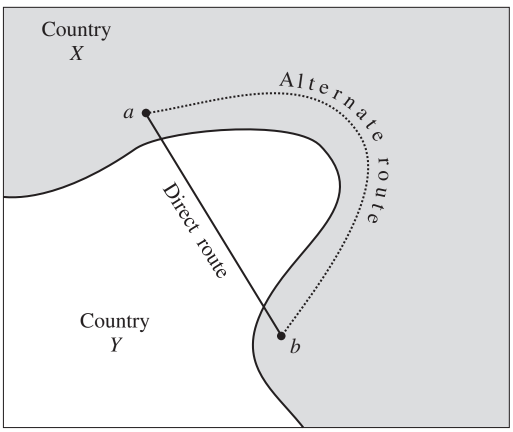

Figure B.1 Use of complex numbers can reduce the work.

Country X), the shortest route is through Country Y, although the journey begins and ends in Country X. We may, if we desire, perform this journey by an alternate route that lies exclusively in X, but this alternate route is longer. In mathematics we have a similar situation with real numbers (Country X) and complex numbers (Country Y). Most real-world problems start with real numbers, and the final results must also be in real numbers. But the derivation of results is considerably simplified by using complex numbers as an intermediary. It is also possible to solve any real-world problem by an alternate method, using real numbers exclusively, but such procedures would increase the work needlessly.

B.1-2 Algebra of Complex Numbers

A complex number (a, b) or $a + jb$ can be represented graphically by a point whose Cartesian coordinates are (a, b) in a complex plane (Fig. B.2). Let us denote this complex number by z so that

$$z = a + jb \tag{B.1}$$

This representation is the Cartesian (or rectangular) form of complex number z. The numbers a and b (the abscissa and the ordinate) of z are the *real part* and the *imaginary part*, respectively, of z. They are also expressed as

$$\operatorname{Re} z = a \qquad \text{and} \qquad \operatorname{Im} z = b$$

Note that in this plane all real numbers lie on the horizontal axis, and all imaginary numbers lie on the vertical axis.

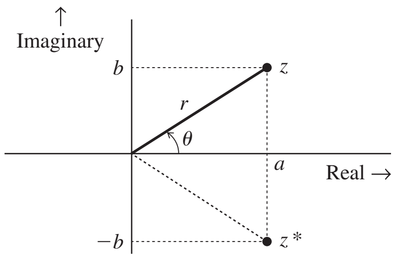

Figure B.2 Representation of a number in the complex plane.

Complex numbers may also be expressed in terms of polar coordinates. If (r, θ) are the polar coordinates of a point $z = a + jb$ (see Fig. B.2), then

$$a = r\cos\theta \qquad \text{and} \qquad b = r\sin\theta$$

Consequently,

$$z = a + jb = r\cos\theta + jr\sin\theta = r(\cos\theta + j\sin\theta) \tag{B.2}$$

Euler's formula states that

$$e^{j\theta} = \cos\theta + j\sin\theta \tag{B.3}$$

To prove Euler's formula, we use a Maclaurin series to expand $e^{j\theta}$, $\cos\theta$, and $\sin\theta$:

$$\begin{aligned}
e^{j\theta} &= 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^5}{5!} + \frac{(j\theta)^6}{6!} + \cdots \\
&= 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - \cdots \\
\cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} \cdots \\
\sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots
\end{aligned}$$

Clearly, it follows that $e^{j\theta} = \cos\theta + j\sin\theta$. Using Eq. (B.3) in Eq. (B.2) yields

$$z = re^{j\theta} \tag{B.4}$$

This representation is the polar form of complex number z.

Summarizing, a complex number can be expressed in rectangular form $a + jb$ or polar form $re^{j\theta}$ with

$$\begin{aligned} a &= r\cos\theta \\ b &= r\sin\theta \end{aligned} \qquad \text{and} \qquad \begin{aligned} r &= \sqrt{a^2+b^2} \\ \theta &= \tan^{-1}\left(\tfrac{b}{a}\right) \end{aligned} \tag{B.5}$$

Observe that r is the distance of the point z from the origin. For this reason, r is also called the *magnitude* (or *absolute value*) of z and is denoted by $|z|$. Similarly, θ is called the angle of z and is denoted by $\angle z$. Therefore, we can also write polar form of Eq. (B.4) as

$$z = |z|e^{j\angle z} \qquad \text{where } |z| = r \text{ and } \angle z = \theta$$

Using polar form, we see that the reciprocal of a complex number is given by

$$\frac{1}{z} = \frac{1}{re^{j\theta}} = \frac{1}{r}e^{-j\theta} = \frac{1}{|z|}e^{-j\angle z}$$

Conjugate of a Complex Number

We define z^*, the *conjugate* of $z = a + jb$, as

$$z^* = a - jb = re^{-j\theta} = |z|e^{-j\angle z} \tag{B.6}$$

The graphical representations of a number z and its conjugate z^* are depicted in Fig. B.2. Observe that z^* is a mirror image of z about the horizontal axis. *To find the conjugate of any number, we need only replace j with $-j$ in that number* (which is the same as changing the sign of its angle).

The sum of a complex number and its conjugate is a real number equal to twice the real part of the number:

$$z + z^* = (a + jb) + (a - jb) = 2a = 2\,\mathrm{Re}\,z$$

Thus, we see that the real part of complex number z can be computed as

$$\mathrm{Re}\,z = \frac{z + z^*}{2} \tag{B.7}$$

Similarly, the imaginary part of complex number z can be computed as

$$\text{Im}\, z = \frac{z - z^*}{2j} \tag{B.8}$$

The product of a complex number z and its conjugate is a real number $|z|^2$, the square of the magnitude of the number:

$$zz^* = |z|e^{j\angle z}|z|e^{-j\angle z} = |z|^2 \tag{B.9}$$

UNDERSTANDING SOME USEFUL IDENTITIES

In a complex plane, $re^{j\theta}$ represents a point at a distance r from the origin and at an angle θ with the horizontal axis, as shown in Fig. B.3a. For example, the number -1 is at a unit distance from the origin and has an angle π or $-\pi$ (more generally, π plus any integer multiple of 2π), as seen from Fig. B.3b. Therefore,

$$-1 = e^{j(\pi+2\pi n)} \qquad n \text{ integer}$$

The number 1, on the other hand, is also at a unit distance from the origin, but has an angle 0 (more generally, 0 plus any integer multiple of 2π). Therefore,

$$1 = e^{j2\pi n} \qquad n \text{ integer} \tag{B.10}$$

The number j is at a unit distance from the origin and its angle is $\frac{\pi}{2}$ (more generally, $\frac{\pi}{2}$ plus any integer multiple of 2π), as seen from Fig. B.3b. Therefore,

$$j = e^{j(\frac{\pi}{2}+2\pi n)} \qquad n \text{ integer}$$

Similarly,

$$-j = e^{j(-\frac{\pi}{2}+2\pi n)} \qquad n \text{ integer}$$

Notice that the angle of any complex number is only known within an integer multiple of 2π.

This discussion shows the usefulness of the graphic picture of $re^{j\theta}$. This picture is also helpful in several other applications. For example, to determine the limit of $e^{(\alpha+j\omega)t}$ as $t \to \infty$, we note that

$$e^{(\alpha+j\omega)t} = e^{\alpha t}e^{j\omega t}$$

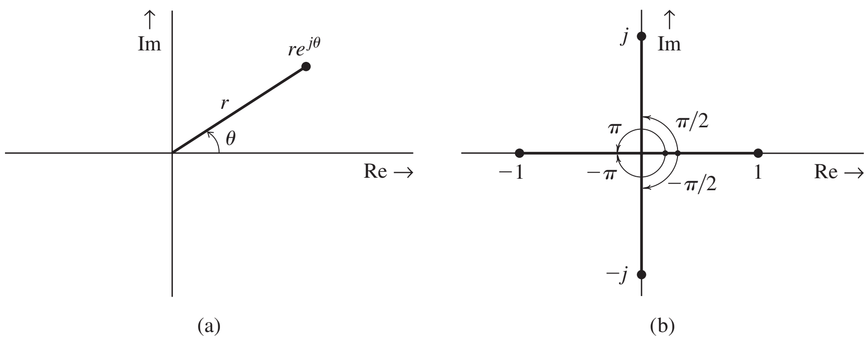

Figure B.3 Understanding some useful identities in terms of $re^{j\theta}$.

Now the magnitude of $e^{j\omega t}$ is unity regardless of the value of ω or t because $e^{j\omega t} = re^{j\theta}$ with $r = 1$. Therefore, $e^{\alpha t}$ determines the behavior of $e^{(\alpha+j\omega)t}$ as $t \to \infty$ and

$$\lim_{t\to\infty} e^{(\alpha+j\omega)t} = \lim_{t\to\infty} e^{\alpha t} e^{j\omega t} = \begin{cases} 0 & \alpha < 0 \\ \infty & \alpha > 0 \end{cases}$$

In future discussions, you will find it very useful to remember $re^{j\theta}$ as a number at a distance r from the origin and at an angle θ with the horizontal axis of the complex plane.

A Warning About Computing Angles with Calculators

From the Cartesian form $a + jb$, we can readily compute the polar form $re^{j\theta}$ [see Eq. (B.5)]. Calculators provide ready conversion of rectangular into polar and vice versa. However, if a calculator computes an angle of a complex number by using an inverse tangent function $\theta = \tan^{-1}(b/a)$, proper attention must be paid to the quadrant in which the number is located. For instance, θ corresponding to the number $-2 - j3$ is $\tan^{-1}(-3/-2)$. This result is not the same as $\tan^{-1}(3/2)$. The former is $-123.7°$, whereas the latter is $56.3°$. A calculator cannot make this distinction and can give a correct answer only for angles in the first and fourth quadrants.† A calculator will read $\tan^{-1}(-3/-2)$ as $\tan^{-1}(3/2)$, which is clearly wrong. When you are computing inverse trigonometric functions, if the angle appears in the second or third quadrant, the answer of the calculator is off by 180°. The correct answer is obtained by adding or subtracting 180° to the value found with the calculator (either adding or subtracting yields the correct answer). For this reason, it is advisable to draw the point in the complex plane and determine the quadrant in which the point lies. This issue will be clarified by the following examples.

EXAMPLE B.1 Cartesian to Polar Form

Express the following numbers in polar form: **(a)** $2+j3$, **(b)** $-2+j1$, **(c)** $-2-j3$, and **(d)** $1-j3$.

(a)

$$|z| = \sqrt{2^2 + 3^2} = \sqrt{13} \qquad \angle z = \tan^{-1}\left(\tfrac{3}{2}\right) = 56.3°$$

In this case the number is in the first quadrant, and a calculator will give the correct value of 56.3°. Therefore (see Fig. B.4a), we can write

$$2 + j3 = \sqrt{13}\, e^{j56.3°}$$

(b)

$$|z| = \sqrt{(-2)^2 + 1^2} = \sqrt{5} \qquad \angle z = \tan^{-1}\left(\tfrac{1}{-2}\right) = 153.4°$$

In this case the angle is in the second quadrant (see Fig. B.4b), and therefore the answer given by the calculator, $\tan^{-1}(1/-2) = -26.6°$, is off by 180°. The correct answer is

† Calculators with two-argument inverse tangent functions will correctly compute angles.

$(-26.6 \pm 180)^\circ = 153.4^\circ$ or -206.6°. Both values are correct because they represent the same angle. It is a common practice to choose an angle whose numerical value is less than 180°. Such a value is called the *principal value* of the angle, which in this case is 153.4°. Therefore,

$$-2 + j1 = \sqrt{5}e^{j153.4^\circ}$$

(c)

$$|z| = \sqrt{(-2)^2 + (-3)^2} = \sqrt{13} \qquad \angle z = \tan^{-1}\left(\tfrac{-3}{-2}\right) = -123.7^\circ$$

In this case the angle appears in the third quadrant (see Fig. B.4c), and therefore the answer obtained by the calculator ($\tan^{-1}(-3/-2) = 56.3^\circ$) is off by 180°. The correct answer is $(56.3 \pm 180)^\circ = 236.3^\circ$ or -123.7°. We choose the principal value -123.7° so that (see Fig. B.4c)

$$-2 - j3 = \sqrt{13}e^{-j123.7^\circ}$$

(d)

$$|z| = \sqrt{1^2 + (-3)^2} = \sqrt{10} \qquad \angle z = \tan^{-1}\left(\tfrac{-3}{1}\right) = -71.6^\circ$$

In this case the angle appears in the fourth quadrant (see Fig. B.4d), and therefore the answer given by the calculator, $\tan^{-1}(-3/1) = -71.6^\circ$, is correct (see Fig. B.4d):

$$1 - j3 = \sqrt{10}e^{-j71.6^\circ}$$

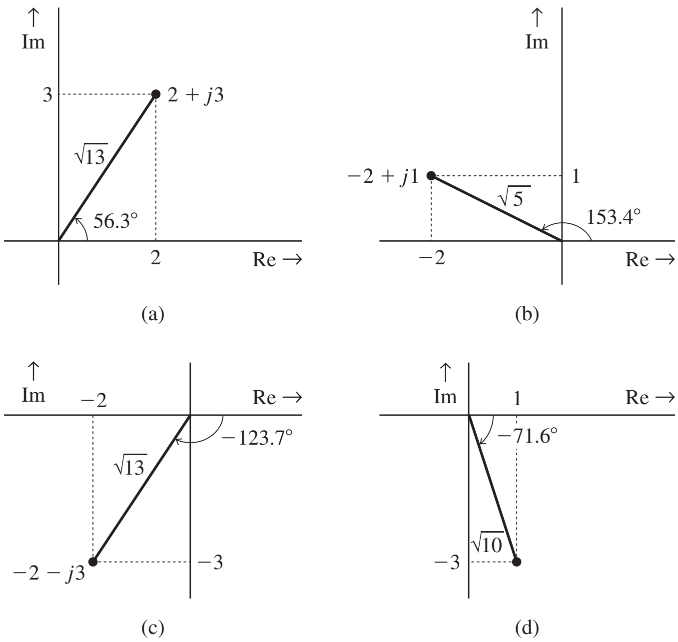

Figure B.4 From Cartesian to polar form.